VARIATION PRINCIPLE AND CONSERVATION EQUATIONS IN NON-LOCAL FIELD THEORY

BY

CLAUDE BLOCH

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Summary.

An interpretation of the non-local fields introduced by H. YUKAWA is given by means of a rule associating any non-local quantity with a corresponding density, which is an ordinary function of space and time. It is shown that, in the spirit of this interpretation, the field equations can be deduced from a variation principle. Conservation equations follow from the invariance properties of the Lagrange function. It appears that a collision process has to be regarded as a whole in accordance with the ideas of the S-matrix theory. Application is made of these principles to the free fields of spin 0 and $1/2$, and the second quantization is introduced. It is shown in particular that, with the proposed interpretation, no physical meaning can be attached to the solutions of the field equations with "internal rotation" described by H. YUKAWA. A Lorentz invariant supplementary equation excluding these extra solutions is established.

I. Introduction.

It has often been suggested in recent years that the divergence difficulties in quantum theory of fields could not be solved without modifying the concepts of conventional field theory for short distances. Recently, H. YUKAWA[1] has proposed a new way of introducing into the theory such a modification by means of non-local fields which are free from the restriction of being point functions in ordinary space-time. The quantities $U^{(i)}$ describing a non-local field are functions both of the four space-time operators x^μ ($x^1 = x_1 = x$, etc. $\cdots x^4 = -x_4 = ct$) and of four space-time displacement operators p_μ, which satisfy the usual commutation relations

$$[x^{\prime\prime}, p_{\nu}] = \delta^{\prime\prime}_{\nu},\tag{1,1}$$

where

$$i[A, B] = AB - BA.\tag{1,2}$$

In a representation in which the operators $x^{\prime\prime}$ are diagonal, the quantities $U^{(i)}$ are described by their matrix elements $(x' \mid U^{(i)} \mid x'')$. (The components of a four-vector x'^1, x'^2, x'^3, x'^4 will usually be denoted by the letter x'; likewise, a four-dimensional volume element $dx'^1 \, dx'^2 \, dx'^3 \, dx'^4$ will be written dx', and $\delta(x')$ will stand for $\prod\limits_{\mu=1}^{4} \delta(x'^\mu)$ each time the argument x' is a four-vector.) Usual local fields, being functions of the space-time operators only, are represented by diagonal matrices such as $u^{(i)}(x') \, \delta(x' - x'')$. They satisfy the commutation relations

$$[U^{(i)}, x^{\prime\prime}] = 0,\tag{1,3}$$

and their matrix elements vanish unless $r^{\prime\prime} = x'^{\prime\prime} - x''^{\prime\prime} = 0$. This is no longer true in the more general case of a non-local field. It is however natural to assume that the matrix elements of $U^{(i)}$ are appreciably different from zero only if all the quantities $r^{\prime\prime}$ are small, say of the order of λ, a length below which the conventional field theory is presumed to require modification. This condition will be expressed by certain equations (Y) which replace (1,3). In the limit $\lambda \to 0$ these equations must require the field to be of the local type. In the case of a free scalar field, for instance, YUKAWA[1] has shown that the equations (1,3) have to be replaced by

$$(Y) \begin{cases} \xi(U) \equiv [[U, x_\mu], x^{\prime\prime}] + \lambda^2 \, U = 0, & (1,4) \\ \eta(U) \equiv [[U, p_\mu], x^{\prime\prime}] \equiv [[U, x^{\prime\prime}], p_\mu] = 0. & (1,5) \end{cases}$$

The field equations can be regarded as a natural generalization of the usual equations[1]; they can also be deduced from a variation principle (cf. II).

The next step is to define the physical quantities connected with the field, such as energy and momentum, angular momentum, and electric charge and current. In the usual theory, these quantities are given by densities in space and time, which are quadratic functions of the field variables. It is therefore neces-

sary to start with a definition of the product of two non-local field variables. We shall take the usual product of operators or matrices since it reduces to the ordinary product in the limiting case of two local quantities. With this definition it will be possible to form expressions generalizing the usual ones for all physical quantities. Being non-local quantities, however, these expressions must be interpreted. A simple way of doing so is to give a rule associating with any non-local quantity A a local quantity $a(x)$ which will be interpreted as an ordinary density in space and time. For this process we require:

1) that, in the limiting case where A becomes a local field,

$$(x' \,|\, A \,|\, x'') \to \alpha(x')\, \delta(x' - x''), \tag{1,6}$$

the associated local density is $\alpha(x)$;

2) that the local density associated with the Hermitian conjugate A^* of A is the complex conjugate of $a(x)$.[1] The correspondence is then given by[2]

$$a(X) = \int dr\, A(X, r), \tag{1,7}$$

where $A(X, r)$ is the matrix element $(x' \,|\, A \,|\, x'')$ expressed in terms of the variables

$$X^\mu = \frac{1}{2}(x'^\mu + x''^\mu), \quad r^\mu = x'^\mu - x''^\mu. \tag{1,8}$$

The integral is taken over the whole space-time r^μ. The conditions 1) and 2) are obviously satisfied since

$$A^*(X, r) = \{A(X, -r)\}^*. \tag{1,9}$$

In section III these considerations will be combined with the variation principle and the invariance conditions of the equations in order to give actual definitions of the physical quantities, following a method very similar to that usually applied to local fields. Special cases of free particles of spin 0 and $\frac{1}{2}$ are dealt with in sections IV and V, and the second quantization is taken into account.

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[1] On this condition $a(x)$ is real if A is Hermitian.
[2] The interpretation by means of such averages has been suggested by Professor C. MØLLER.

II. The variation principle.

In the conventional field theory the field equations are generally deduced from a variation principle, a procedure which is particularly convenient in view of the fact that it leads to a general definition of quantities satisfying conservation equations which can be identified with energy, momentum etc. In the case of non-local fields, we assume the existence of a non-local Hermitian Lagrange function $\mathcal{L}$, depending on the field variables and the operators x^μ, p_μ, which in the limiting case of a local field reduces to $L(X)\,\delta(r)$, where $L(X)$ is the ordinary local Lagrange function, and $\delta(r) = \prod_{\mu=1}^{4} \delta(r^\mu)$. The simplest expressions which can be formed by means of the field variables and the operators x^μ and p_μ are the linear functions of

$$\left.\begin{array}{ll} \text{a) } [U^{(i)}, p_\mu], & \text{b) } [U^{(i)}, x^\mu], \\[2mm] \text{c) } [U^{(i)}, p_\mu]_+, & \text{d) } [U^{(i)}, x^\mu]_+, \end{array}\right\} \tag{2,1}$$

where

$$[A, B]_+ = AB + BA. \tag{2,2}$$

In the limit $U^{(i)} \to u^{(i)}(X)\,\delta(r)$, these expressions have the limits

$$\left.\begin{array}{ll} \dfrac{\partial u^{(i)}(X)}{\partial X^\mu}\,\delta(r), & ir^\mu u^{(i)}(X)\,\delta(r), \\[4mm] -2\,iu^{(i)}(X)\dfrac{\partial\delta(r)}{\partial r^\mu}, & 2\,X^\mu u^{(i)}(X)\,\delta(r), \end{array}\right\} \tag{2,3}$$

respectively. The expression (d) can be disregarded if space-time is supposed to be homogeneous, (a) alone is sufficient to generalize in a formal way all usual Lagrange functions, (b) and (c) could be necessary in order to introduce new features into the theory. For the moment, let us assume that $\mathcal{L}$ is a polynomial in the variables $U^{(i)}$ and $[U^{(i)}, p_\mu]$, and try to extend the variation principle.

From $\mathcal{L}$ we define a local density according to (1,7) by

$$L(X) = \int dr\,\mathcal{L}(X, r). \tag{2,4}$$

Let Ω be the part of the space-time contained in a three-dimensional closed hypersurface Σ, and consider the integral

$$I_\Omega = \int_\Omega dX\, L(X).\tag{2,5}$$

For the sake of simplicity, it is convenient to introduce the operator J_Ω defined by $(x'|J_\Omega|x'') = 1$ if the point $X^\mu = {}^1\!/_2 (x'^\mu + x''^\mu)$ is inside Ω, and $(x'|J_\Omega|x'') = 0$ if X^μ is outside Ω. The integral I_Ω can then be written

$$I_\Omega = Tr\, J_\Omega \mathcal{L} = Tr\, \mathcal{L} J_\Omega,\tag{2,6}$$

where Tr is the trace operation defined by

$$Tr\, A = \int dx'\, (x'|A|x').\tag{2,7}$$

Use will be made below of the operator $[p_\mu, J_\Omega]$. From the formula

$$Tr\, A\,[B, C] = Tr\, B\,[C, A] = Tr\, C\,[A, B],\tag{2,8}$$

(which follows at once from $Tr\, AB = Tr\, BA$), we obtain

$$Tr\, J_\Omega\,[A, p_\mu] = Tr\,[p_\mu, J_\Omega]\, A.$$

Now, by a partial integration,

$$Tr\, J_\Omega\,[A, p_\mu] = \int_\Omega dX \int dr\, \frac{\partial A(X, r)}{\partial X^\mu} = \int_\Sigma d\sigma_\mu \int dr\, A,$$

where $d\sigma_\mu$ is the three-dimensional surface element on Σ. Hence, the meaning of $[p_\mu, J_\Omega]$ is given by

$$Tr\,[p_\mu, J_\Omega]\, A = \int_\Sigma d\sigma_\mu \int dr\, A;\tag{2,9}$$

it could also be obtained by direct interpretation.

Let now $\delta U^{(i)}$ be an infinitesimal variation of the field variables. From the assumptions made on $\mathcal{L}$ it follows that

$$\mathcal{L}\delta = \sum_{m, i, \mu} \mathcal{L}^m_{(i)1}\, \delta U^{(i)}\, \mathcal{L}^m_{(i)2} + \mathcal{L}^{\mu, m}_{(i)1}\, [\delta U^{(i)}, p_\mu]\, \mathcal{L}^{\mu, m}_{(i)2},\tag{2,10}$$

where the quantities $\mathfrak{L}$ are functions of the field variables gene-
ralizing the partial derivatives. Considering the variation of
$I_\Omega, \delta I_\Omega = Tr\, J_\Omega\, \delta\mathfrak{L}$, we can, using (2,8), perform the following
transformations on the terms containing $[\delta U^{(i)}, p_\mu]$ (the indices
i, μ and m have been omitted for the $\mathfrak{L}$'s).

$$Tr\, J_\Omega\, \mathfrak{L}_1 [\delta U^{(i)}, p_\mu]\, \mathfrak{L}_2 = Tr\, \mathfrak{L}_2\, J_\Omega\, \mathfrak{L}_1 [\delta U^{(i)}, p_\mu] =$$

$$= Tr\, [p_\mu, \mathfrak{L}_2\, J_\Omega\, \mathfrak{L}_1]\, \delta U^{(i)} =$$

$$= Tr\, \{[p_\mu, \mathfrak{L}_2]\, J_\Omega\, \mathfrak{L}_1 + \mathfrak{L}_2\, J_\Omega\, [p_\mu, \mathfrak{L}_1]\}\, \delta U^{(i)} + Tr\, \mathfrak{L}_2\, [p_\mu, J_\Omega]\, \mathfrak{L}_1\, \delta U^{(i)}.$$

Hence, we get $\delta I_\Omega = \delta I_1 + \delta I_\Sigma$, where

$$\delta I_1 = Tr \sum_i \mathfrak{F}_{(i)}\, \delta U^{(i)}, \qquad\qquad (2,11)$$

$$\delta I_\Sigma = Tr\, \mathfrak{L}_{(i)2}^{\mu, m} [p_\mu, J_\Omega]\, \mathfrak{L}_{(i)1}^{\mu, m}\, \delta U^{(i)}, \qquad\qquad (2,12)$$

the definition of $\mathfrak{F}_{(i)}$ being

$$\mathfrak{F}_{(i)} = \sum_{m, \mu} \mathfrak{L}_{(i)2}^{m}\, J_\Omega\, \mathfrak{L}_{(i)1}^{m} - [\mathfrak{L}_{(i)2}^{\mu, m}, p_\mu]\, J_\Omega\, \mathfrak{L}_{(i)1}^{\mu, m} -$$
$$- \mathfrak{L}_{(i)2}^{\mu, m}\, J_\Omega\, [\mathfrak{L}_{(i)1}^{\mu, m}, p_\mu]. \qquad\qquad (2,13)$$

From the interpretation of J_Ω and $[p_\mu, J_\Omega]$ it is seen that δI_1
is an integral extended to the volume Ω, whereas δI_Σ is a sur-
face integral over Σ. In the usual field theory, the variation
principle states that δI_Ω must vanish for any variation $\delta U^{(i)}$
vanishing on the boundary Σ of the volume of integration. This
boundary condition has no precise meaning in the theory of
non-local fields. Strictly speaking, a surface integral such as
δI_Σ may depend on the values of $(x' | \delta U^{(i)} | x'')$ for all values
of x'^μ and x''^μ. In fact, the field variables satisfy the equations
(Y), and it will be shown below that also the variations $\delta U^{(i)}$
have to satisfy the equations (Y). It follows that the non-local
quantities $U^{(i)}, \delta U^{(i)}$ and $\mathfrak{L}$ have matrix elements appreciably dif-
ferent from zero only when $x'^\mu \sim x''^\mu$, the sign $\sim$ standing for an
approximate equality with an error of the order of λ. Consider
now a term in (2,12); it may be written explicitly

$$\int dx'' \, (x'' \, | \, \mathfrak{L}_2 \, | \, x''') \, dx''' \, (x''' \, | \, [p_\mu, J_\Omega] \, | \, x'''') \, dx'''' \, (x'''' \, | \, \mathfrak{L}_1 \, | \, x') \, dx' \, (x' \, | \, \delta \, U^{(i)} \, | \, x'').$$

The differential element of the integral is large only if $x'''^\mu \sim x''^\mu \sim x'^\mu \sim x''''^\mu$, and if $\frac{1}{2} (x'''^\mu + x''''^\mu)$ are the coordinates of a point on Σ. This shows that practically the integral depends only on the values of $(x' \, | \, \delta \, U^{(i)} \, | \, x'')$, where $x'^\mu \sim x''^\mu \sim X^\mu$, and X^μ are the coordinates of a point on Σ. It appears now that the boundary condition must be that $(x' \, | \, \delta \, U^{(i)} \, | \, x'')$ vanishes if x'^μ and x''^μ are within a *neighbourhood* of Σ of the dimension λ. This cannot be formulated rigorously except in the limiting case in which Ω is the whole space-time. The boundary condition becomes then the usual general condition of vanishing rapidly enough at infinity. However, if Ω is not infinite, but has dimensions very large compared with λ, we shall assume that, as a consequence of the boundary condition for $\delta U^{(i)}$, we still have rigorously $\delta I_\Sigma = 0$. If the variations $\delta U^{(i)}$ are not submitted to any other condition, the variation principle $\delta I_\Omega = \delta I_1 = 0$ gives the field equations

$$\mathfrak{F}_{(i)} = 0. \tag{2,14}$$

As we have not taken equations (Y) into account, these equations and the field equations $(2,14)$ might be incompatible[1]. In order to get consistent equations, the variations $\delta U^{(i)}$ must be restricted to those compatible with (Y). If the equations (Y) are linear, the conditions are

$$\xi \, (\delta \, U^{(i)}) = 0, \quad \eta \, (\delta \, U^{(i)}) = 0. \tag{2,15}$$

The method to be used now is the well-known method of the Lagrangian multipliers[2]. In our case, these are non-local quantities $V_{(i)}$ and $W_{(i)}$ by means of which we form the expression

$$\delta I' = Tr \sum_i \left\{ V_{(i)} \, \xi \, (\delta \, U^{(i)}) + W_{(i)} \, \eta \, (\delta \, U^{(i)}) \right\}, \tag{2,16}$$

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[1] The number of equations obtained from a variation principle is equal to the number of components of the field. Thus, one cannot expect from the variation principle to get the field equations and also the equations (Y). The latter equations have to be postulated or to be deduced from other considerations. Alternatively, one could require $\mathfrak{L}$ to be chosen such that the equations $(2,14)$ are compatible with the equations (Y). However, it does not seem easy to draw any general conclusion from this condition.

and we get the field equations from $\delta I_\Omega + \delta I' = 0$ for *arbitrary* variations $\delta U^{(i)}$. If the equations (Y), for instance, are precisely the equations $(1,4)$ and $(1,5)$, the expression $(2,16)$ is readily transformed into

$$\delta I' = Tr \sum_i \{\xi(V_{(i)}) + \eta(W_{(i)})\} \, \delta U^{(i)}. \tag{2,17}$$

Thus, the field equations are now, instead of $(2,14)$,

$$\mathcal{S}_{(i)} + \xi(V_{(i)}) + \eta(W_{(i)}) = 0. \tag{2,18}$$

After introduction of the Lagrangian multipliers the number of unknown functions becomes equal to the number of equations. The new variables introduced in this way can be interpreted as a kind of field necessary to maintain the "internal structure" of the particles defined by the equations (Y).

Another new feature of the theory should be emphasized. In the usual field theory, if we divide the domain Ω into two parts, $\Omega = \Omega_1 + \Omega_2$, we get at once, from a solution of the variation problem in Ω, a solution in Ω_1 and a solution in Ω_2. Conversely, it is possible to build up a solution in Ω from a solution in Ω_1 and a solution in Ω_2, if we choose the proper boundary values for the field variables on the common border of Ω_1 and Ω_2. This results from the possibility of treating the problems in Ω_1 and Ω_2 quite independently. In the theory of non-local fields, the solution of the variation problem in any domain, strictly speaking, involves the knowledge of the field functions in the whole space-time. Thus, the problems related to Ω_1 and Ω_2 overlap in such a way that it may be impossible to find a solution valid simultaneously in Ω_1 and Ω_2. We must then expect a dependence of the solutions on the volume and, in fact, it is seen from $(2,13)$ that the field equations $(2,18)$ depend explicitly on Ω. Of course, this dependence must become negligible as soon as the dimensions of Ω are taken very much larger than λ, and each physical quantity must have a limiting value if $\Omega \to \infty$. In fact, the overlapping is appreciable only in a neighbourhood of the common border of Ω_1 and Ω_2 of the dimension λ; accordingly, the dependence on Ω will be very small if the dimensions of Ω are much larger than λ. More precisely, it will be seen that free non-local fields behave in this respect rigorously

like local fields, and from this it follows that, if Ω is so large that the interactions between the various fields are negligible on the boundary Σ, the solution of the field equations does not depend appreciably on Ω.

III. The conservation equations.

The conservation equations can be deduced from the variation principle and the invariance properties of the Lagrange function[3]. From now on, it will be asssumed that the field functions $U^{(i)}$ are solutions of the field equations and of the equations (Y). Furthermore, the considered variations $\delta U^{(i)}$ will always be compatible with the equations (Y), i. e. satisfy the equations (2,15). Hence $\delta I' = 0$, $\delta I_1 = 0$, and

$$\delta I_\Omega = \delta I_\Sigma. \tag{3,1}$$

The variations $\delta U^{(i)}$ will be the variations due to an infinitesimal change of the frame of reference defined by

$$\bar{x}^u = x^u - \delta x^u, \quad \bar{p}_\mu = p_\mu - \delta p_\mu. \tag{3,2}$$

For any quantity A the variation at a fixed "point" is defined by

$$\delta^* A = \bar{A}(\bar{x}, \bar{p}) - A(x, p), \tag{3,3}$$

and the variation at constant "coordinates", which will be the variation introduced in (3,1), by

$$\delta A = \bar{A}(x, p) - A(x, p). \tag{3,4}$$

Between the two variations holds the relation

$$\delta A - \delta^* A = \left(\delta x'^u \frac{\partial}{\partial x'^u} + \delta x''^u \frac{\partial}{\partial x''^u} \right) (x' | A | x'') =$$
$$= [A, \delta x^u p_\mu]. \tag{3,5}$$

For any scalar invariant quantity, $\delta^* A = 0$. In particular, the Lagrange function must be invariant against changes of coordinates. Hence $\delta^* \mathcal{L} = 0$, and

$$\delta \pounds = [\pounds, \delta x^\mu p_\mu]. \tag{3,6}$$

The variation δI_Ω is then

$$\delta I_\Omega = Tr\, J_\Omega\, \delta\pounds = Tr\, J_\Omega\, [\pounds, \delta x^\mu p_\mu] = Tr\, [\delta x^\mu p_\mu, J_\Omega]\, \pounds, \tag{3,7}$$

with the help of the identity (2,8). It is readily transformed into[1]

$$\delta I_\Omega = \frac{1}{2}\, Tr\, [p_\mu, J_\Omega]\, [\delta x^\mu, \pounds]_+. \tag{3,8}$$

Finally, we get from (2,12), (3,1) and (3,8)

$$Tr\, [p_\mu, J_\Omega] \left\{ \mathfrak{M}^\mu(\delta U^{(i)}) - \frac{1}{2}\, [\delta x^\mu, \pounds]_+ \right\} = 0, \tag{3,9}$$

where

$$\mathfrak{M}^\mu(\delta U^{(i)}) = \sum_{i,m} \pounds^{\mu,m}_{(i)1}\, \delta U^{(i)}\, \pounds^{\mu,m}_{(i)2}. \tag{3,10}$$

The equation (3,9) involves only a surface integral over Σ; it is the general conservation equation. According to (2,9), it can also be written

$$\int_\Sigma d\sigma_\mu \int dr \left\{ \mathfrak{M}^\mu(\delta U^{(i)}) - \delta X^\mu \pounds \right\} = 0. \tag{3,11}$$

The application of the equation (3,11) to the special cases of infinitesimal translations or Lorentz transformations leads to conservation equations corresponding to energy-momentum and to angular momentum.

A. Energy-momentum.

An infinitesimal translation is defined by

$$\delta x_\mu = \varepsilon_\mu, \quad \delta p_\mu = 0. \tag{3,12}$$

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[1] From the definition of J_Ω it follows that it anticommutes with p_μ; indeed $[J_\Omega \cdot p_\mu]_+ = -i\frac{\partial}{\partial r^\mu} J_\Omega = 0$. It is seen from the formulas given below that for an infinitesimal translation or Lorentz transformation holds $p_\mu\, \delta x^\mu = \delta x^\mu p_\mu$. We have then

$$i\left[\delta x^\mu p_\mu, J_\Omega\right] = \delta x^\mu p_\mu J_\Omega - J_\Omega p_\mu \delta x^\mu = \frac{i}{2}\left\{\delta x^\mu\left[p_\mu, J_\Omega\right] + \left[p_\mu, J_\Omega\right]\delta x^\mu\right\},$$

from which (3,8) follows immediately.

We have $\delta^* U^{(i)} = 0$, and from (3,5)

$$\delta U^{(i)} = \varepsilon_\mu [U^{(i)}, p^\mu]. \tag{3,13}$$

The equation (3,11) gives the conservation equation

$$\int_\Sigma d\sigma_\nu \int dr \, T^{\mu\nu} = 0, \tag{3,14}$$

where

$$T^{\mu\nu} = \mathfrak{R}^\nu ([U^{(i)}, p^\mu]) - g^{\mu\nu} \mathfrak{L} \tag{3,15}$$

can be interpreted as the canonical energy-momentum tensor of the field ($g^{\mu\nu}$ is the metric tensor $g^{11} = g^{22} = g^{33} = 1$, $g^{44} = -1$, $g^{\mu\nu} = 0$ if $\mu \neq \nu$).

B. Angular momentum.

An infinitesimal Lorentz transformation is given by

$$\delta x_\mu = \varepsilon_{\mu\nu} x^\nu, \quad \delta p_\mu = \varepsilon_{\mu\nu} p^\nu, \quad \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0. \tag{3,16}$$

The variation at a fixed point of the field components depends on the vectorial character of the field. In a general way it can be written

$$\delta^* U^{(i)} = \frac{1}{2} S_j^{i\lambda\mu} U^{(j)} \varepsilon_{\lambda\mu}; \tag{3,17}$$

hence, from (3,5)

$$\delta U^{(i)} = \frac{1}{2} \left\{ S_j^{i\lambda\mu} U^{(j)} - [U^{(i)}, \omega^{\lambda\mu}] \right\} \varepsilon_{\lambda\mu}, \tag{3,18}$$

where

$$\omega^{\lambda\mu} = x^\lambda p^\mu - p^\lambda x^\mu. \tag{3,19}$$

We get now from (3,11) the conservation equation

$$\int_\Sigma d\sigma_\nu \int dr \, M^{\lambda\mu\nu} = 0, \tag{3,20}$$

in which

$$M^{\lambda\mu\nu} = \mathfrak{R}^\nu ([U^{(i)}, \omega^{\lambda\mu}]) - (x^\lambda g^{\mu\nu} - x^\mu g^{\lambda\nu}) \mathfrak{L} - \mathfrak{R}^\nu (S_j^{i\lambda\mu} U^{(j)}) \tag{3,21}$$

is the angular momentum tensor of the system.

C. Electric charge and current.

Now we assume that the field variables $U^{(i)}$ are complex. The Lagrange function is then a function of $U^{*(i)}$ as well as of $U^{(i)}$, the corresponding $\mathfrak{M}^\mu$'s being functions of $\delta U^{(i)}$ and $\delta U^{*(i)}$. The Lagrange function is supposed to be invariant against the gauge transformation

$$U^{(i)} \to e^{i\alpha} U^{(i)}, \quad U^{*(i)} \to U^{*(i)} e^{-i\alpha}, \tag{3,22}$$

where α is an arbitrary constant. Consider an infinitesimal gauge transformation given by

$$\delta U^{(i)} = i\delta\alpha \, U^{(i)}, \quad \delta U^{*(i)} = -i \, U^{*(i)} \delta\alpha; \tag{3,23}$$

we have $\delta\mathfrak{L} = 0$. Hence (3,1) reduces to $\delta I_\Sigma = 0$, or

$$\int_\Sigma d\sigma_\nu \int dr \, J^\nu = 0, \tag{3,24}$$

where

$$J^\nu = i\varepsilon \, \mathfrak{M}^\nu (U^{(i)}, -U^{*(i)}) \tag{3,25}$$

represents the electric current-charge of the system.

The densities in space-time of energy, momentum, angular momentum, electric charge and current are defined, according to the rule (1,7), by

$$t^{\mu\nu} = \int dr \, T^{\mu\nu}, \quad m^{\lambda\mu\nu} = \int dr \, M^{\lambda\mu\nu}, \quad j^\nu = \int dr \, J^\nu; \tag{3,26}$$

they satisfy the conservation equations

$$\int_\Sigma d\sigma_\nu \, t^{\mu\nu} = 0, \quad \int_\Sigma d\sigma_\nu \, m^{\lambda\mu\nu} = 0, \quad \int_\Sigma d\sigma_\nu \, j^\nu = 0. \tag{3,27}$$

However, as the surface Σ is not arbitrary, they do not satisfy the usual continuity equations

$$\frac{\partial \, t^{\mu\nu}}{\partial \, X^\nu} = 0, \quad \frac{\partial \, m^{\lambda\mu\nu}}{\partial \, X^\nu} = 0, \quad \frac{\partial \, j^\nu}{\partial \, X^\nu} = 0, \tag{3,28}$$

which will be valid only in the special cases where the field satisfies the equations (2,18) with any *arbitrary* domain Ω. Suppose now that Ω is the domain enclosed between two space-like surfaces, $\sigma_{(1)}$ being very far in the past and $\sigma_{(2)}$ very far in the future. Consider

$$G^\mu = \int_\sigma d\sigma_\nu \, t^{\mu\nu}, \quad P^{\lambda\mu} = \int_\sigma d\sigma_\nu m^{\lambda\mu\nu}, \quad Q = \int_\sigma d\sigma_\nu j^\nu, \quad (\sigma = \sigma_{(1)} \text{ or } \sigma_{(2)}), \quad (3,29)$$

where the orientation of the surface element $d\sigma_\nu$ is such that $d\sigma_4 > 0$; from (3,27) it follows that the integrals (3,29) *have the same values on $\sigma_{(1)}$ and on $\sigma_{(2)}$*. This means that the quantities defined by (3,29) are constants of the collision, although we cannot follow the conservation in detail during the process. Consequently, collision problems in non-local field theory will have to be treated by means of a formalism similar to the S-matrix theory.

According to BELINFANTE and ROSENFELD[4], it is possible to define a tensor $\theta^{\mu\nu} = \int dr \, \Theta^{\mu\nu}$ satisfying the conditions

$$G^\mu = \int_\sigma d\sigma_\nu \theta^{\mu\nu}, \quad P^{\lambda\mu} = \int_\sigma d\sigma^\nu (X^\lambda \, \theta^{\mu\nu} - X^\mu \, \theta^{\lambda\nu}), \qquad (3,30)$$

where σ is $\sigma_{(1)}$ or $\sigma_{(2)}$. The expressions (3,30) will in fact be equal to the expressions (3,29) if two non-local tensors $F^{\mu\nu\sigma}$ and $H^{\lambda\mu\nu\sigma}$ exist such that

$$\left. \begin{aligned} \Theta^{\mu\nu} - T^{\mu\nu} &= \frac{\partial F^{\mu\nu\sigma}}{\partial X^\sigma}, &\qquad F^{\mu\nu\sigma} + F^{\mu\sigma\nu} &= 0, \\[2mm] X^\lambda \, \Theta^{\mu\nu} - X^\mu \, \Theta^{\lambda\nu} - M^{\lambda\mu\nu} &= \frac{\partial H^{\lambda\mu\nu\sigma}}{\partial X^\sigma}, &\qquad H^{\lambda\mu\nu\sigma} + H^{\lambda\mu\sigma\nu} &= 0. \end{aligned} \right\} \quad (3,31)$$

Putting

$$S^{\lambda\mu\nu} = X^\lambda \, T^{\mu\nu} - X^\mu \, T^{\lambda\nu} - M^{\lambda\mu\nu}, \qquad (3,32)$$

we get from (3,31)

$$\frac{\partial H^{\lambda\mu\nu\sigma}}{\partial X^\sigma} = S^{\lambda\mu\nu} - F^{\lambda\mu\nu} + F^{\mu\lambda\nu} + \frac{\partial}{\partial X^\sigma} (X^\lambda \, F^{\mu\nu\sigma} - X^\mu \, F^{\lambda\nu\sigma}).$$

Thus, the problem is solved on defining $F^{\lambda\mu\nu}$ by

$$S^{\lambda\mu\nu} = F^{\lambda\mu\nu} - F^{\mu\lambda\nu}, \quad F^{\lambda\mu\nu} + F^{\lambda\nu\mu} = 0,$$

hence

$$2 \, F^{\lambda\mu\nu} = S^{\lambda\mu\nu} - S^{\mu\nu\lambda} + S^{\nu\lambda\mu}, \qquad (3,33)$$

and then $\Theta^{\mu\nu}$ by the first equation (3,31). The tensor $\Theta^{\mu\nu}$ is not symmetric in general, but $\theta^{\mu\nu}$ *is symmetric in the cases where the continuity equations* (3,28) *hold*.

IV. The free scalar field.

In every theory of a free field, the plane wave plays an essential part. It is convenient as non-local plane wave to introduce

$$\text{Exp}\,(k,l) = \exp\left(\frac{i}{2}kx\right)\exp\,(ilp)\exp\left(\frac{i}{2}kx\right) \left.\begin{array}{c} \\ \\ \end{array}\right\} (4,1)$$
$$= \exp\left(\frac{i}{2}lp\right)\exp\,(ikx)\exp\left(\frac{i}{2}lp\right).$$

(The scalar product $a^\mu b_\mu$ is denoted by ab, and the ordinary exponential function by exp; in (4,1), k and l are of course four-vectors). The function (4,1) is chosen in such a way that

$$\text{Exp}^*\,(k,l) = \text{Exp}\,(-k,-l). \qquad (4,2)$$

The product of two functions Exp is given by

$$\text{Exp}\,(k,l)\,\text{Exp}\,(k',l') = \text{Exp}\,(k+k',l+l')\exp\frac{i}{2}(k'l-kl'), \quad (4,3)$$

and, corresponding to the derivatives, we have the commutation relations

$$[\text{Exp}\,(k,l),p_\mu] = ik_\mu\,\text{Exp}\,(k,l), \left.\begin{array}{c} \\ \\ \end{array}\right\} (4,4)$$
$$[\text{Exp}\,(k,l),x^\mu] = -il^\mu\,\text{Exp}\,(k,l).$$

The matrix elements of Exp are

$$(x'\,|\,\text{Exp}\,(k,l)\,|\,x'') = \delta\,(r+l)\exp\,(ik\,X), \qquad (4,5)$$

from which it follows that

$$Tr\,\text{Exp}\,(k,l) = (2\,\pi)^4\,\delta\,(k)\,\delta\,(l). \qquad (4,6)$$

From (4,5) it is seen that any arbitrary non-local quantity A can be expanded as

$$A = \int dk\,dl\,a\,(k,l)\,\text{Exp}\,(k,l). \qquad (4,7)$$

The inversion formula of (4,7) is readily found with the help of (4,2), (4,3), and (4,6)

$$a\,(k,l) = (2\,\pi)^{-4}\,TrA\,\text{Exp}^*\,(k,l). \qquad (4,8)$$

In particular, the expansion of J_Ω is

$$J_\Omega = \int dk\, dl\, \delta_\Omega(k)\, \mathrm{Exp}(k, l), \tag{4,9}$$

where

$$\delta_\Omega(k) = (2\,\pi)^{-4} \int_\Omega dX \exp(-ikX) \tag{4,10}$$

reduces to the ordinary δ function if Ω is the whole space-time.

A. Field equations.

The simplest Hermitian expression which in the limiting case of a local field reduces to the usual Lagrange function of a complex scalar field is

$$\mathfrak{L} = [U^*, p^\mu][U, p_\mu] + \varkappa^2\, U^* U, \tag{4,11}$$

where $\varkappa$ is $c/\hbar$ times the rest mass of the particles.

From (2,13) and (2,18) we get the field equation

$$\{[[U, p_\mu], p^\mu] - \varkappa^2\, U\}\, J_\Omega = \xi(V) + \eta(W) \tag{4,12}$$

and the conjugate of the latter. Further, in the present case, (3,10) gives

$$\mathfrak{M}^\nu = [U^*, p^\nu]\, \delta U + \delta U^*\, [U, p^\nu]; \tag{4,13}$$

hence, we get for the energy-momentum, the angular momentum tensors, and the electric current-charge four-vector

$$\left.\begin{aligned}
T^{\mu\nu} &= [U^*, p^\mu][U, p^\nu] + [U^*, p^\nu][U, p^\mu] - g^{\mu\nu}\,\mathfrak{L}, \\
M^{\lambda\mu\nu} &= [U^*, \omega^{\lambda\mu}][U, p^\nu] + [U^*, p^\nu][U, \omega^{\lambda\mu}] - \{X^\lambda\, g^{\mu\nu} - X^\mu\, g^{\lambda\nu}\}\,\mathfrak{L}, \\
J^\nu &= i\varepsilon\{[U^*, p^\nu]\, U - U^*\, [U, p^\nu]\}.
\end{aligned}\right\} \tag{4,14}$$

We shall first investigate the field equations (4,12) in the case where Ω is the whole space-time. We call J the corresponding operator J_Ω. The field U can be represented by an expansion of the type (4,7) which, taking into account the equations (Y), (1,4) and (1,5), may be written

$$U = \int dk\, dl\, u(k, l)\, \delta(kl)\, \delta(l^2 - \lambda^2)\, \mathrm{Exp}(k, l). \tag{4,15}$$

The Lagrangian multipliers V and W can also be expanded according to (4,7). Let $v(k, l)$ and $w(k, l)$ be the corresponding

coefficients in the expansion. Replacing in (4,12) all the functions, including J, by their expansions we get after a short calculation

$$(k^2 + \varkappa^2) \int dl'\, u\,(k,\, l')\, \delta\,(kl')\, \delta\,(l'^2 - \lambda^2) =$$
$$= \{(l^2 - \lambda^2)\, v\,(k,\, l) + kl\, w\,(k,\, l)\}\, \exp\left(\frac{i}{2}\, kl\right). \qquad (4,16)$$

The left-hand side does not depend on l. Thus, the value of the right-hand side is not changed if we multiply it by $\varkappa/2\,\pi\lambda\,\delta\,(kl)\,\delta\,(l^2 - \lambda^2)\, dl$ and integrate over all values of l. The result is clearly zero. Hence $\xi\,(V) + \eta\,(W) = 0$, and as V and W appear in the equations only through this expression, we may take them equal to zero. We are left with

$$(k^2 + \varkappa^2) \int dl'\, u\,(k,\, l')\, \delta\,(kl')\, \delta\,(l'^2 - \lambda^2) = 0. \qquad (4,17)$$

To interpret this equation we split $u\,(k,\, l)$ into two terms[1]

$$u\,(k,\, l) = u_0\,(k) + u_1\,(k,\, l), \qquad (4,18)$$

in such a way that

$$\int dl\, u_1\,(k,\, l)\, \delta\,(kl)\, \delta\,(l^2 - \lambda^2) = 0. \qquad (4,19)$$

The equation (4,17) becomes now

$$(k^2 + \varkappa^2)\, u_0\,(k) = 0,$$

which shows that $u_0\,(k)$ has to be of the form

$$u_0\,(k) = \bar{u}\,(k)\, \delta\,(k^2 + \varkappa^2). \qquad (4,20)$$

It may seem strange at first sight that, apart from the condition (4,19), $u_1\,(k,\, l)$ remains entirely arbitrary. This is due to the fact that, if we compute the integral (2,5) and replace U by its expansion (4,15) in $\mathfrak{L}$, writing $u\,(k,\, l)$ as in (4,18), no contribution comes from the term $u_1\,(k,\, l)$. Furthermore, the energy-momentum or electric current-charge *local densities* (3,26), corresponding to a wave of the type

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[1] The term $u_1\,(k,\, l)$ corresponds to the set of states of the particles with "internal rotation", described by YUKAWA.

$$\int dl\, u_1\,(k,\,l)\,\delta\,(kl)\,\delta\,(l^2-\lambda^2)\,\mathrm{Exp}\,(k,\,l),$$

are vanishing. The reason is that all the preceding expressions involve integrals of the type

$$\int dr\, dl\, dl'\, a^*\,(k,\,l)\,b\,(k,\,l')\,\delta\,(kl)\,\delta\,(l^2-\lambda^2)\,\delta\,(kl')\,\delta\,(l'^2-\lambda^2)\,\mathrm{Exp}^*\,(k,\,l)\,\mathrm{Exp}\,(k,\,l') =$$
$$= \int dl\, a^*\,(k,\,l)\,\delta\,(kl)\,\delta\,(l^2-\lambda^2)\int dl'\, b\,(k,\,l')\,\delta\,(kl')\,\delta\,(l'^2-\lambda^2),$$

which vanish as soon as one of the terms a or b is a u_1 term. It seems then natural to assume that no physical meaning should be attached to the part u_1 of the field and to suppress these components by adding as a *supplementary equation*

$$u_1\,(k,\,l) = 0. \tag{4,21}$$

The equation (4,21) has to be Lorentz invariant and compatible with the other equations. This is obviously true in the present case. It is now readily seen that the field satisfies the following equation given by YUKAWA

$$[[U,\,p_\mu],\,p^\mu] - \varkappa^2\, U = 0, \tag{4,22}$$

hence it satisfies the equation (4,12) for any *arbitrary* domain Ω, and the continuity equations (3,28) hold.

The condition (4,21) can be transformed with the help of the inversion formula (4,8) into an equation involving U directly of the form

$$(Y')\quad U\,(1) = Tr_{(2)}\, P\,(1,\,2)\, U\,(2), \tag{4,23}$$

where 1 and 2 indicate that the functions depend on two different sets of variables x^μ, p_μ, commuting with each other, denoted by 1 and 2. The operator

$$P\,(1,\,2) = (2\,\pi)^{-4}\int dk\, dl\, \mathrm{Exp}_0\,(k,\,1)\,\mathrm{Exp}^*\,(k,\,l,\,2), \tag{4,24}$$

where we have put

$$\mathrm{Exp}_0\,(k) = \frac{\varkappa}{2\,\pi\lambda}\int dl\,\delta\,(kl)\,\delta\,(l^2-\lambda^2)\,\mathrm{Exp}\,(k,\,l), \tag{4,25}$$

is a projection operator. It satisfies indeed the characteristic relation of projectors, which here reads

$$Tr_{(2)} P(1,2) P(2,3) = P(1,3). \qquad (4,26)$$

Let the unit operator $I(1,2)$ be defined by

$$A(1) = Tr_{(2)} I(1,2) A(2), \quad \text{for any } A; \qquad (4,27)$$

it is seen from (4,7) and (4,8) that

$$I(1,2) = (2\pi)^{-4} \int dk \, dl \, \mathrm{Exp}(k,l,1) \, \mathrm{Exp}^*(k,l,2). \qquad (4,28)$$

Denoting by $Q(1,2)$ the projector $I(1,2) - P(1,2)$, we can write (4,23) in the form

$$(Y') \qquad Tr_{(2)} Q(1,2) U(2) = 0. \qquad (4,29)$$

It should be noted that the equations (Y') contain the equations (Y). If we now replace the equations (Y) by (Y') in the derivation of the field equations, the expression (2,16) has to be replaced by

$$\delta I' = Tr_{(1,2)} V(2) Q(2,1) \, \delta U(1) = 0, \qquad (4,30)$$

where V is again a Lagrangian multiplier. Thus, the field equations become

$$\mathfrak{S}(1) + Tr_{(2)} V(2) Q(2,1) = 0. \qquad (4,31)$$

In the special case of a free field, from the equation (4,31) results of course

$$\mathfrak{S} = 0, \quad Tr_{(2)} V(2) Q(2,1) = 0.$$

The general solution of the field equations and of the equation (Y') is given by the expansion

$$U = \int dk \, \bar{u}(k) \, \delta(k^2 + \varkappa^2) \, \mathrm{Exp}_0(k). \qquad (4,32)$$

If we define, as usually, the local density associated with U by

$$u(X) = \int dr \, U(X, r), \qquad (4,33)$$

it is easily seen with the help of the expansion (4,32) that conversely the field U is given in terms of $u(X)$ by the formula

$$U = (2\pi)^{-4} \int dX \, dk \, u(X) \exp(-ik\,X) \, \mathrm{Exp}_0(k). \qquad (4,34)$$

This shows that the generality of the non-local field has been reduced by the equation (Y'') so that its number of degrees of freedom has become equal to that of an ordinary local field. Note that it follows from (4,34) or (4,23) that $\tilde{U} = U$, where $\tilde{U}$ is the transposed operator of U defined by $(x' | \tilde{U} | x'') = (x'' | U | x')$.

B. Quantization.

So far the theory is equivalent to a c-numbers field theory. The second quantization has now to be performed. We start from the local field $u(X)$ defined by (4,33). It satisfies the usual wave equation

$$\frac{\partial^2 u}{\partial X^\mu \partial X_\mu} - \varkappa^2 u = 0. \tag{4,35}$$

Furthermore, $u(X)$ may be any solution of (4,35) since the formula (4,34) gives a non-local field satisfying the field equation (4,22) as soon as $u(X)$ satisfies (4,35). Hence, the commutation relations of $u(X)$ must be identical with the usual[3] commutation relations

$$[u(X_{(1)}), \ u(X_{(2)})] = [u^*(X_{(1)}), u^*(X_{(2)})] = 0,$$
$$[u(X_{(1)}), u^*(X_{(2)})] = [u^*(X_{(1)}), u(X_{(2)})] = -\hbar D(X_{(1)} - X_{(2)}), \tag{4,36}$$

where the function D is

$$D(X) = i(2\pi)^{-3} \int dk \, \varepsilon(k) \, \delta(k^2 + \varkappa^2) \exp(ik X), \tag{4,37}$$

in which $\varepsilon(k)$ is $+1$ or -1, depending on whether k^4 is positive or negative. The formula (4,34) gives now the commutation relations of the non-local field itself[1]

$$[U(1), \ U(2)] = [U^*(1), U^*(2)] = 0,$$
$$[U(1), U^*(2)] = [U^*(1), \ U(2)] = -\hbar \mathfrak{D}(1,2), \tag{4,38}$$

where

$$\mathfrak{D}(1,2) = i(2\pi)^{-3} \int dk \, \varepsilon(k) \, \delta(k^2 + \varkappa^2) \operatorname{Exp}_0(k, 1) \operatorname{Exp}_0^*(k, 2). \tag{4,39}$$

[1] These commutation relations are different from those which can be deduced from Yukawa's paper[1]. In fact, Yukawa's commutation relations are not compatible with the equation (Y') which should then be considered a supplementary condition, whereas it can be taken as an operator equation if the function $\mathfrak{D}(1,2)$ is defined according to (4,39).

The commutation relations (4,38) are clearly Lorentz inva-
riant and compatible with the field equations and with equation
(Y'). This completes the covariant formulation of the basic equa-
tions. However, it is interesting to give a physical picture of
the system by investigating its stationary states. As Lorentz
invariance is not very important here, we shall assume that
the field regarded as a function of X^1, X^2, X^3 has periods equal
to a length L very much larger than λ. Then the effects of non-
localizability in the neighbourhood of the boundaries of the cube
can be neglected. The expansion (4,32) becomes a series

$$U = \hbar^{\frac{1}{2}} L^{-\frac{3}{2}} \sum_k (2\,k^4)^{-\frac{1}{2}} \{a_k \operatorname{Exp}_0(k) + b_k^* \operatorname{Exp}_0^*(k)\}, \quad (4,40)$$

where

$$k^1 = \frac{2\,\pi}{L}\,n_1, \quad k^2 = \frac{2\,\pi}{L}\,n_2, \quad k^3 = \frac{2\,\pi}{L}\,n^3, \quad k^4 = +\sqrt{\underline{k}^2 + \varkappa^2}. \quad (4,41)$$

The numerical factor in front of (4,40) has been taken such
that the commutation relations for the amplitudes a_k and b_k
resulting from (4,36) or (4,38) are

$$i\,[a_k,\,a_{k'}^*] = \delta_{kk'}, \quad i\,[b_k,\,b_{k'}^*] = \delta_{kk'}, \quad (4,42)$$

the other commutators being zero. Thus, the operators

$$n_k^+ = a_k^*\,a_k, \quad n_k^- = b_k^*\,b_k \quad (4,43)$$

have the eigenvalues $0, 1, 2, \cdots$ Replacing U by the expansion
(4,40) in the expressions (3,29), where the integration is now
extended to the cube L^3, we get

$$\left.\begin{aligned}
G^\mu &= \sum_k \hbar k'^\mu\,(n_k^+ + n_k^- + 1), \\
Q &= \sum_k \hbar\,\varepsilon\,(n_k^+ - n_k^- - 1),
\end{aligned}\right\} \quad (4,44)$$

which shows that the system consists of n_k^+ particles with the
energy-momentum $\hbar k^\mu$ and the charge $\hbar\varepsilon$, and of n_k^- particles
with the energy-momentum $\hbar k^\mu$ and the charge $-\hbar\varepsilon$. It is pos-
sible to get rid of the zero-point charge by replacing at the starting
point $\mathcal{L}$ by $\frac{1}{2}(\mathcal{L} + \tilde{\mathcal{L}})$. The spin of the particles is clearly zero;

the expectation value of $P^{\lambda\mu}$ vanishes in fact in the state in which there is only one particle with momentum zero. Finally, it should be noted that, although $T^{\mu\nu}$ is symmetric, it is not identical with $\Theta^{\mu\nu}$ defined by the equations (3,30).

V. The free electron field.

A. Field equations.

In Dirac's theory of the electron four matrices γ^μ occur, which satisfy the relations

$$[\gamma^\mu, \gamma^\nu]_+ = 2\, g^{\mu\nu}. \tag{5,1}$$

The matrices $\gamma^1, \gamma^2, \gamma^3$ are Hermitian, γ^4 is antihermitian. We define a Lagrange function by

$$2\mathcal{L} = \frac{1}{2}\{\Psi^+ \gamma^\mu [\Psi, p_\mu] - [\Psi^+, p_\mu]\gamma^\mu \Psi\} + \varkappa\,\Psi^+\Psi \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} \tag{5,2}$$
$$+ \frac{1}{2}\{\Psi'^+ \gamma^\mu [\Psi', p_\mu] - [\Psi'^+, p_\mu]\gamma^\mu \Psi'\} + \varkappa\,\Psi'^+\Psi',$$

in which $\Psi^+ = i\,\Psi^* \gamma^4$ is the adjoint field function, and $\Psi' = C\,\tilde{\Psi}^+$, $\Psi'^+ = C^{-1}\,\tilde{\Psi}$ are the charge conjugate field functions. The matrix C is the usual unitary skew-symmetric matrix such that

$$\tilde{\gamma}^\mu = -C^{-1}\gamma^\mu C. \tag{5,3}$$

The expression (5,2) is Hermitian and is invariant against changes of coordinates, gauge transformations

$$\begin{aligned}\Psi &\to e^{i\alpha}\,\Psi, & \Psi^+ &\to \Psi^+ e^{-i\alpha}, \\ \Psi' &\to e^{-i\alpha}\,\Psi', & \Psi'^+ &\to \Psi'^+ e^{i\alpha},\end{aligned} \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} \tag{5,4}$$

and charge conjugation. As regards the equations which have to restrict the non-localizability of the field functions, the simplest assumption is that each component of Ψ has to satisfy the equations (1,4) and (1,5), i. e.

$$(Y) \quad \xi(\Psi) = 0, \quad \eta(\Psi) = 0. \tag{5,5}$$

Similar equations hold then for the adjoint and the charge con-

jugate field functions. These equations clearly satisfy all inva-
riance requirements.

Considering, for the moment, the field functions and the
charge conjugate field functions as independent from each other,
we get from the variation principle the field equations

$$\left.\begin{aligned} \{\gamma^{\mu}\,[\Psi, p_{\mu}] + \varkappa\,\Psi\}\,J_{\Omega} &= \xi\,(V) + \eta\,(W), \\ J_{\Omega}\,\{-[\Psi^{+}, p_{\mu}]\,\gamma^{\mu} + \varkappa\,\Psi^{+}\} &= \xi\,(V^{+}) + \eta\,(W^{+}), \end{aligned}\right\} \quad (5,6)$$

and for the charge conjugate functions

$$\left.\begin{aligned} \{\gamma^{\mu}\,[\Psi', p_{\mu}] + \varkappa\,\Psi'\}\,J_{\Omega} &= \xi\,(V') - \eta\,(W'), \\ J_{\Omega}\,\{-[\Psi'^{+}, p_{\mu}]\,\gamma^{\mu} + \varkappa\,\Psi'^{+}\} &= \xi\,(V'^{+}) - \eta\,(W'^{+}). \end{aligned}\right\} \quad (5,7)$$

On the other hand,

$$4\,\mathfrak{S}\mathfrak{X}^{\nu} = \delta\Psi^{+}\,\gamma^{\nu}\,\Psi - \Psi^{+}\,\gamma^{\nu}\,\delta\Psi + \text{charge conjugate terms}; \quad (5,8)$$

hence,

$$\left.\begin{aligned} T^{\mu\nu} &= \frac{1}{4}\,\{[\Psi^{+}, p^{\mu}]\,\gamma^{\nu}\,\Psi - \Psi^{+}\,\gamma^{\nu}\,[\Psi, p^{\mu}] + \text{ch. conj.}\} - g^{\mu\nu}\,\mathfrak{L}, \\ M^{\lambda\mu\nu} &= \frac{1}{4}\,\{[\Psi^{+}, \omega^{\lambda\mu}]\,\gamma^{\nu}\,\Psi - \Psi^{+}\,\gamma^{\nu}\,[\Psi, \omega^{\lambda\mu}] - \Psi^{+}\,\gamma^{\lambda}\,\gamma^{\mu}\,\gamma^{\nu}\,\Psi + \text{ch. conj.}\} \\ &\quad - \{X^{\lambda}\,g^{\mu\nu} - X^{\mu}\,g^{\lambda\nu}\}\,\mathfrak{L}, \\ J^{\nu} &= i\,\varepsilon\,\{\Psi^{+}\,\gamma^{\nu}\,\Psi - \text{ch. conj.}\}, \end{aligned}\right\} \quad (5,9)$$

since, as is well-known,

$$\delta^{*}\,\Psi = -\frac{1}{4}\,\varepsilon_{\lambda\mu}\,\gamma^{\lambda}\,\gamma^{\mu}\,\Psi, \quad \delta^{*}\,\Psi^{+} = \frac{1}{4}\,\varepsilon_{\lambda\mu}\,\Psi^{+}\,\gamma^{\lambda}\,\gamma^{\mu}, \text{etc.} \cdots \quad (5,10)$$

As in the case of the scalar field, we investigate first the case
in which Ω is the whole space-time. Taking into account the
equations (5,5), Ψ can be expanded as

$$\Psi = \int dk\, dl\, \psi\,(k, l)\,\delta\,(kl)\,\delta\,(l^{2} - \lambda^{2})\,\mathrm{Exp}\,(k, l), \quad (5,11)$$

in which $\psi\,(k, l)$ is now a spinor. Replacing Ψ, and similarly
V and W in (5,6) by their expansions, it is seen that the argu-

ment used for the scalar field can be applied to each component of the present equations; it yields

$$\xi(V) + \eta(W) = 0. \tag{5,12}$$

Putting

$$\psi(k, l) = \psi_0(k) + \psi_1(k, l), \tag{5,13}$$

where

$$\int dl\, \psi_1(k, l)\, \delta(kl)\, \delta(l^2 - \lambda^2) = 0, \tag{5,14}$$

the field equations read

$$(ik_\mu \gamma^\mu + \varkappa)\, \psi_0(k) = 0. \tag{5,15}$$

Similar equations hold for the adjoint and charge conjugate functions. Again we notice that a pure ψ_1 plane wave has zero *local densities* of energy-momentum and electric current-charge; accordingly we put as a *supplementary equation*

$$\psi_1(k, l) = 0 \tag{5,16}$$

or

$$(Y') \quad Tr_{(2)}\, Q(1, 2)\, \Psi(2) = 0, \tag{5,17}$$

where $Q(1, 2)$ is the same projection operator as in (4,29). The equation (5,17) contains the equations (5,5). As a consequence of equation (5,17) Ψ satisfies

$$\gamma^\mu [\Psi, p_\mu] + \varkappa \Psi = 0; \tag{5,18}$$

hence, the equations (5,6) and (5,7) are satisfied for any arbitrary Ω; accordingly, the continuity equations (3,28) hold. The local field function associated with Ψ being defined by

$$\psi(X) = \int dr\, \Psi(X, r), \tag{5,19}$$

Ψ is given by the inversion formula

$$\Psi = (2\pi)^{-4} \int dX\, dk\, \psi(X) \exp(-ikX)\, Exp_0(k). \tag{5,20}$$

Note that from this expression it follows that $\tilde{\Psi} = \Psi$.

B. Quantization.

Again we start from the field $\psi(X)$ which, being a solution of the ordinary Dirac equations, must also satisfy the usual bracket relations[3]

$$[\psi(X_{(1)}), \psi(X_{(2)})]_+ = [\psi^+(X_{(1)}), \psi^+(X_{(2)})]_+ = 0, \left.\right\}$$
$$i[\psi_\varrho(X_{(1)}), \psi_\sigma^+(X_{(2)})]_+ = \hbar\left\{-\gamma^\mu \frac{\partial}{\partial X_{(1)}^\mu} + \varkappa\right\}_{\varrho\sigma} D(X_{(1)} - X_{(2)}), \left.\right\} \quad (5,21)$$

where $D(X)$ is the function defined by (4,37). Formula (5,20) gives now the bracket relations for the non-local quantities

$$[\Psi(1), \Psi(2)]_+ = [\Psi^+(1), \Psi^+(2)]_+ = 0, \left.\right\}$$
$$i[\Psi_\varrho(1), \Psi_\sigma^+(2)]_+ = \hbar\left\{-\gamma_{\varrho\sigma}^\mu[\mathfrak{D}(1,2), p_{(1)\mu}] + \varkappa\,\delta_{\varrho\sigma}\,\mathfrak{D}(1,2)\right\}, \left.\right\} \quad (5,22)$$

where $\mathfrak{D}(1,2)$ is the function defined by (4,39). The same relations hold for the charge conjugate functions, and the whole system clearly satisfies all the invariance requirements.

Now we consider the field enclosed inside a cube L^3 with the same boundary conditions as for the scalar field. The expansion of Ψ becomes a series

$$\Psi = \hbar^{\frac{1}{2}} L^{-\frac{3}{2}} \sum_{k,r} \{a_k^r\,\psi_+^r(k)\,\mathrm{Exp}_0(k) + b_k^{*r}\,\psi_-^r(k)\,\mathrm{Exp}_0^*(k)\}, \quad (5,23)$$

in which the spinors $\psi_+^r(k)$ and $\psi_-^r(k)$, $(r = 1, 2)$ are solutions of the equations (5,15) corresponding to k_μ and $-k_\mu$, respectively, and orthonormalized according to

$$\psi_+^{*r}(k)\,\psi_+^s(k) = \delta^{rs}, \quad \psi_-^{*r}(k)\,\psi_-^s(k)\,\delta^{rs}. \quad (5,24)$$

It should be recalled that, in expansion (5,23), $k^4 > 0$. The bracket relations for the a_k^r and b_k^r resulting from (5,21) or (5,22) are

$$[a_k^r, a_k^{*r}]_+ = [b_k^r, b_k^{*r}]_+ = 1, \quad (5,25)$$

all other anticommutators being zero. It follows that the operators

$$n_{+k}^r = a_k^{*r}\,a_k^r, \quad n_{-k}^r = b_k^{*r}\,b_k^r \quad (5,26)$$

have the eigenvalues 0 and 1. Replacing now Ψ by the expansion (5,23) in the expressions (3,29), and extending the integration to the cube L^3, we get

$$G^{\mu} = \sum_{k,r} \hbar k^{\mu} (n^{r}_{+k} + n^{r}_{-k} - 1),$$

$$Q = \sum_{k,r} \hbar \varepsilon (n^{r}_{+k} - n^{r}_{-k}),$$

$$\left.\right\} \quad (5,27)$$

the interpretation of which in terms of particles is again obvious.

VI. Conclusion.

An attempt has been made to develop the non-local field theory along lines following as closely as possible the general scheme of the conventional field theory. The whole procedure of the non-quantized field theory has been extended to the non-local fields. In the simple cases of the free fields of spin 0 and $\frac{1}{2}$, it gives consistent results and, in particular, it does not lead to the introduction of any new degree of freedom. The quantization can easily be introduced for free fields. However, it has still to be extended to the case of several fields in interaction, for which the present theory gives the field equations only. Finally, it should be noted that, for the Lagrange functions as well as for the equations (Y), only the simplest expressions have been considered. The theory, being a generalization of the usual theory, offers new possibilities which, however, cannot be thoroughly investigated before the interactions of the various fields have been treated in greater detail.

References.

(1) H. Yukawa. Phys. Rev. **76**, 300 (1949); **76**, 1731, (1949); **77**, 219, (1950); I am grateful to Professor H. Yukawa for having placed the last mentioned manuscript at the disposal of the Institute previous to publication.

(2) Courant and Hilbert, Die Methoden der mathematischen Physik, 2nd ed. I, p. 189.

(3) See, for instance, W. Pauli, Rev. Mod. Phys. **13**, 203, (1941).

(4) F. J. Belinfante, Physica **6**, 887, (1939); **7**, 305, (1940); L. Rosenfeld, Mém. Acad. Roy. Belg. **6**, 30, (1940).